Substitution Method

$f_0 = 0 \quad f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

recurrence relation

$a_1 = 3 \qquad a_n = a_{n-1} + 3n - 1$ for $n \geq 2$

$a_2 = 8$
$a_3 = 16$

So $a_n = a_{n-1} + 3n - 1 \qquad a_{n-1} = a_{n-2} + 3(n-1) - 1$
$= a_{n-2} + 3(n-1) - 1 + 3n - 1 \qquad a_{n-2} = a_{n-3} + 3(n-2) - 1$
$= a_{n-3} + 3(n-2) - 1 + 3(n-1) - 1 + 3n - 1$
$\vdots$
$= a_2 + 8 + \cdots + 3n - 1 \qquad a_3 = a_2 + 8$
$= a_1 + 5 + 8 + \cdots + 3n - 1$
$= 3 + \underbrace{5}_{i=2} + 8 + \cdots + \underbrace{3n-1}_{i=n}$
$= 3 + \sum_{i=2}^{n} 3i - 1 \qquad \underbrace{1}_{i=2} + 1 + \cdots + \underbrace{1}_{i=n}$
$= 3 + \sum_{i=2}^{n} 3i - \sum_{i=2}^{n} 1 \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
$= 3 + 3\sum_{i=2}^{n} i - (n-1) \qquad n=3 \quad \frac{27+3+2}{2} = 16$
$= 3 + 3\left(\frac{n(n+1)}{2} - 1\right) - (n-1) = \frac{3n^2+n+2}{2}$

$c_1 = 2 \qquad c_n = 5c_{n-1} + 1$ for $n \geq 2$

$c_2 = 11$
$c_3 = 56$

$c_n = 5^1 c_{n-1} + 1 \qquad c_{n-1} = 5 \cdot c_{n-2} + 1$
$= 5(5 \cdot c_{n-2} + 1) + 1$
5^2
$= 25 \cdot c_{n-2} + 5 + 1 \qquad c_{n-2} = 5 \cdot c_{n-3} + 1$
$1 = n - x$
$= 25 \cdot (5 \cdot c_{n-3} + 1) + 5 + 1$
$x = n-1$
$c_1 = c_{n-(n-1)}$
5^3
$= 125 \cdot c_{n-3} + 25 + 5 + 1 \qquad \sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1}$
$\vdots \qquad r = 5 \quad k = n-2$
$= 5^{n-1} \cdot c_1$
$= 5^{n-1} c_{n-(n-1)} + \sum_{i=0}^{n-2} 5^i$
$= 2 \cdot 5^{n-1} + \frac{5^{n-1} - 1}{4} = \frac{9 \cdot 5^{n-1} - 1}{4} \qquad c_3 = \frac{9 \cdot 25 - 1}{4} = \frac{224}{4} = 56$

$b_1 = 4 \qquad b_n = 3b_{n-1} + n$

$\frac{d}{dr} \sum_{i=0}^{k} r^i = \frac{d}{dr} \frac{r^{k+1} - 1}{r - 1} \qquad \sum_{i=0}^{k} i \cdot r^{i-1} =$

$$\frac{d}{dr} \sum_{i=0} \quad \frac{\overline{}}{r-1} \Big/ dr \quad \sum_{i=0} i \cdot r^{i-1} =$$

Constructive Induction

$d_0 = 0$
$d_1 = 6$

for $n \geq 2$
$d_n = 8 d_{n-1} - 15 d_{n-2}$
A B

$f_n = 1 \cdot f_{n-1} + 1 \cdot f_{n-2}$
A B

2nd order linear homogeneous recurrence relation w/ constant coefficients

guess $d_n = r^n$

better guess $d_n = C \cdot r^n + D \cdot s^n$
where r, s are solns to

$s_n = A \cdot s_{n-1} + B \cdot s_{n-2}$
where $B \neq 0$

Ind step: Suppose $k \geq 2$ and $d_i = r^i$ for $0 \leq i < k$
[want $d_k = r^k$]

$d_n = 3 \cdot 5^n - 3 \cdot 3^n$

Then $d_k = 8 d_{k-1} - 15 \cdot d_{k-2}$

$0 \leq k-1, k-2 < 2$
b/c $k \geq 2$
so ind. hyp. applies

$= 8 \cdot r^{k-1} - 15 \cdot r^{k-2}$

$d_0 = C \cdot 5^0 + D \cdot 3^0$
$0 = C + D$

[want $r^k = 8 \cdot r^{k-1} - 15 \cdot r^{k-2}$

$r^2 = 8r - 15$
$r^2 - 8r + 15 = 0$]
$(r-5)(r-3)$
$r = 5$ or $r = 3$

$d_1 = C \cdot 5^1 + D \cdot 3$
$6 = 5C + 3D$
$C = 3 \quad D = -3$

$r^2 - Ar - B$

$d_n = C \cdot r^n + D \cdot s^n$
if two distinct roots

$= C \cdot r^n + D \cdot n \cdot r^n$
if one root

$\sum_{i=0}^{n} i^2 = an^3 + bn^2 + cn + d$ for $n \geq 0$

Proof: Base case $(n=0)$: $0 = a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d \rightarrow d = 0$

Ind. step: Suppose $n \geq 0$ and $\sum_{i=0}^{n} i^2 = an^3 + bn^2 + cn$

[want: $\sum_{i=0}^{n+1} i^2 = a(n+1)^3 + b(n+1)^2 + c(n+1)$
$= an^3 + (3a+b)n^2 + (3a+2b+c)n + (a+b+c)$]

Then $\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 + (n+1)^2$

$1 + 4 + 9 + 16 = 30$

$= an^3 + bn^2 + cn + n^2 + 2n + 1$
$= an^3 + (b+1)n^2 + (c+2)n + 1$

want $=$

for these 2 poly to $=$, need coeffs $=$

so $a = a$
$3a + b = b + 1 \rightarrow a = \frac{1}{3}$
$3a + 2b + c = c + 2 \rightarrow b = \frac{1}{2}$
$1 = a + b + c \rightarrow c = \frac{1}{6}$

$\sum_{i=0}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$
$= \frac{2n^3 + 3n^2 + n}{6}$
$= \frac{n(2n+1)(n+1)}{6}$

$\frac{4 \cdot 9 \cdot 5}{6} = 30$

empty

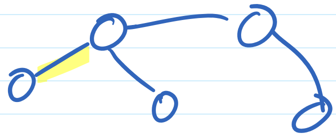
— is legal

if s is legal, so is (s)

if s_1, s_2 are legal, so is $s_1 s_2$

$\underbrace{(())}_{s_1}\underbrace{()}_{s_2}$

$\forall n \in \mathbb{Z}, n \geq n, P(n)$



THM: For all legal strings s, open(s) = closed(s)

(#)

For all $n \geq 1$, for all legal s built by applying rules n times, open(s) = closed(s)

Proof: Base case (n=1): Any legal s built w/ 1 rule is built with base case only, so is empty. The empty string has 0 open, 0 close

Ind step: Suppose $k \geq 2$, s is legal and built w/ k rules, and all legal s' built w/ i applications of rules, $1 \leq i < k$ have open(s') = closed(s')

Then, since s is built with $k \geq 2$ rules, is built by applying one of 2 recursive cases last

Case 1: s = (s') for some legal s'

s' is ~~built~~ ~~w/~~ i ~~rules~~ ~~for~~ $1 \leq i < k$; so ind. hyp. applies to s'

$\therefore$ open(s') = close(s')

open(s) = 1 + open(s') = 1 + close(s') = close(s)

case 2: $s = s_1 s_2$ for some legal $s_1 s_2$

s_1, s_2 ~~built~~ ~~w/~~ i_1, i_2 ~~rules~~ ~~for~~ $1 \leq i < k$ so ind. hyp applies to s_1, s_2

So open(s_1) = close(s_1) and open(s_2) = close(s_2)

and open(s) = open(s_1) + open(s_2)
= close(s_1) + close(s_2) = close(s)

structural induction omits the parts related to the transformation to "for all $n \geq 1$, legal s built w/ n rules ..."