Pierogies

$\forall k \geq a,\ \underbrace{P(k)}_{} \to P(k+1)$

base case

Yocco's sells pierogi in orders of 3, 4, and 7.

Can you order exactly	5	pierogies?	N
	6	3+3	Y
$23 = 4+4+4+4+4+3$	7	7	Y
$24 = 4+4+4+4+4+4$	8	4+4	Y
$25 = 3+3+3+4+4+4+4$	9	3+3+3	Y
$28 = 7+7+7+7$	10	3+7	Y
$29 = 4+4+7+7+7$			⋮

\# of orders of 3, of 4, of 7

$\{0,1,2,\ldots\}$

THM: For all integers $n \geq 6$, $\underbrace{\exists a,b,c \in \mathbb{N} \text{ s.t. } n = 3a+4b+7c.}_{P(n)}$

Proof: Base case: $6 = 3\cdot\underset{a}{2} + 4\cdot\underset{b}{0} + 7\cdot\underset{c}{0}$

Ind. step: Suppose $k \geq 6$ and $\exists a,b,c \in \mathbb{N}$ s.t. $k = 3a+4b+7c$; find them

[want $\exists a',b',c'$ s.t. $k+1 = 3a'+4b'+7c'$ and $a',b',c' \in \mathbb{N}$]

Then $a \geq 1$ or $b \geq 2$ or $c \geq 1$ since otherwise $a=0$ and $b \leq 1$ and $c=0$ and so

$k = 3a+4b+7c = 4b \leq 4$ contradicting $k \geq 6$ $\Rightarrow\Leftarrow$

3 cases: 1) $a \geq 1$:
$$3(a-1)+4(b+1)+7c = 3a-3+4b+4+7c = 3a+4b+7c+1 = k+1$$

So $\exists a',b',c' \in \mathbb{N}$ s.t. $k+1 = 3a'+4b'+7c$ (namely $a'=a-1$, $b'=b+1$, $c'=c$)

so $a' \geq 0$
so $a' \in \mathbb{N}$

$b-2 \geq 2-2 = 0$

2) $b \geq 2$
$$3(a+3)+4(b-2)+7c = 3a+4b+7c+9-8 = k+1$$

So $\exists a',b',c' \in \mathbb{N}$ s.t. $k+1 = 3a'+4b'+7c'$ (namely $a'=a+3$, $b'=b-2$, $c'=c$)

c) $c \geq 1$
$$3a+4(b+2)+7(c-1) = 3a+4b+7c+8-7 = k+1$$

So $\exists a',b',c' \in \mathbb{N}$ s.t. $k+1 = 3a'+4b'+7c'$ (namely $a'=a$, $b'=b+2$, $c'=c-1$)

Let $k \in \mathbb{Z}$
Suppose $G(k) \wedge P(k)$
Suppose $\sim p \wedge \sim q \wedge \sim r$
⋮
$\sim G(k)$
$G(k) \wedge \sim G(k)$
c
$(\sim p \wedge \sim q \wedge \sim r) \to c$
$\therefore \sim(\sim p \wedge \sim q \wedge \sim r) \equiv p \vee q \vee r$
Suppose p
⋮
$P(k+1)$
$\therefore p \to P(k+1)$
Suppose q
⋮
$P(k+1)$
$\therefore q \to P(k+1)$
Suppose r
⋮
$P(k+1)$
$\therefore r \to P(k+1)$
$\therefore P(k+1)$
$\therefore G(k) \wedge P(k) \to P(k+1)$
$\therefore \forall k \in \mathbb{Z},\ G(k) \wedge P(k) \to P(k+1)$

Inequalities

THM: $\forall n \in \mathbb{Z},\ n \geq 6 \rightarrow n^3 - 6 \geq 5n^2 + 10$

Proof:

Sequences

$a_1 = 7$
$a_2 = 2\cdot 7^2 + 7 + 2 = 107$
$a_3 = 2\cdot 107^2 + 107 + 2 = \ldots$

Define a sequence $a_1, a_2, \ldots$ by $a_1 = 7$ and $\underline{a_n = 2a_{n-1}^2 + a_{n-1} + 2}$ for $n \geq 2$.

THM: For all integers $n \geq 1$, $\underbrace{a_n \equiv 2 \pmod 5}_{P(n)}$.

Proof: Base case $(n=1)$ $a_1 = 7$ $7 \equiv 2 \pmod 5$ b/c $5 \mid 7-2$ $P(k+1)$

Ind step: Suppose $k \geq 1$ and $\overbrace{a_k \equiv 2 \pmod 5}^{P(k)}$ [want $a_{k+1} \equiv_5 2$]

Then by def of seq a, $a_{k+1} = 2\cdot a_k^2 + a_k + 2$

$\rightarrow$ (b/c $a_k \equiv 2 \pmod 5$) by ind. hyp.

$\equiv 2\cdot 2^2 + 2 + 2 \pmod 5$

$\equiv 12 \pmod 5$

$\equiv 2 \pmod 5$ $5 \mid 12-2$

Fibonacci

Define a sequence $a_0, a_1, \ldots$ by $a_0 = 0$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
	0	1	1	2	3	5	8	13
2^n	1	2	4	8	16	32	64	128

THM: For all $n \geq 0$, $\overbrace{a_n \leq 2^n}^{P(n)}$

Proof: Base case: $(n=0)$ $a_0 = 0$ $2^0 = 1$ $0 \leq 1$

$a_{k-1} \leq 2^{k-1}$

Ind step: Suppose $k \geq 0$ and $\overbrace{a_k \leq 2^k}^{P(k)}$ and $a_{k-1} \leq 2^{k-1}$ $P(k+1)$ [want $a_{k+1} \leq 2^{k+1}$]

Then $a_{k+1} = a_k + a_{k-1}$

$\leq 2^k + \underline{a_{k-1}}$

Strong Induction

$P(a)$
$P(a+1)$ base cases
$\vdots$
$P(b)$

$P(a) \wedge P(a+1) \wedge \cdots \wedge P(b) \to P(b+1)$ $\therefore P(b+1)$
$P(a) \wedge P(a+1) \wedge \cdots \wedge P(b+1) \to P(b+2)$ $\therefore P(b+2)$
$P(a) \wedge P(a+1) \wedge \cdots \wedge P(b+2) \to P(b+3)$ $\therefore P(b+3)$
$\vdots$

inductive hypothesis

$\forall k \in \mathbb{Z} \left(k > b \wedge \forall i \in \mathbb{Z}, a \le i \le k-1 \to P(i) \right) \to P(k)$

Suppose $k \in \mathbb{Z}$ s.t. $k > b$ and $P(i)$ for any $a \le i \le k-1$

(want $P(k)$)

Define a sequence $a_0, a_1, \ldots$ by $a_0 = 0$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \ge 2$.

THM: For all $n \ge 0$, $\overbrace{a_n \le 2^n}^{P(n)}$

Proof: Base cases: $(n=0)$: $a_0 = 0$ $2^0 = 1$ $0 \le 1$ ✓

$(n=1)$: $a_1 = 1$ $2^1 = 2$ $1 \le 2$ ✓

so need $k-2 \ge 0$, $k \ge 2$

need $k-1$, $k-2$ in range

Ind step: Suppose $k > 1$ and $\overbrace{a_i \le 2^i}^{P(i)}$ for all $i \in \mathbb{Z}$ s.t. $0 \le i < k$

(want $\underbrace{a_k \le 2^k}_{P(k)}$)

Then $a_k = a_{k-1} + a_{k-2}$
$\le 2^{k-1} + 2^{k-2}$ (by ind. hypoth since $k > 1$ guarantees $0 \le k-1, k-2 < 1$)
$= \frac{1}{2} \cdot 2^k + \frac{1}{4} 2^k = \frac{3}{4} 2^k \le 2^k$

Define sequence $b_0, b_1, \ldots$ by $b_0 = 0$, $b_1 = 1$, $b_n = b_{n-1} + b_{n-2} + 1$ for $n \ge 2$

THM: For all integers $n \ge 0$, $b_n \le 3a_n$

Proof: Base cases

Ind step:

THM: For all integers $n \geq 1$,

Base cases:

Ind. step: Let $k > 2$ and suppose $b_i \leq 3a_i - 1$ for $1 \leq i < k$

Then $b_k = b_{k-1} + b_{k-2} + 1$