Arguments with Quantifiers

D = set of students
FF(x) = "x likes free food"

$p \vee q$
$\sim p$
$\therefore q$

All students like free food.
Chun-wai is a student
∴ Chun-wai likes free food

$\forall x \in D, FF(x)$
Chun-wai $\in D$

universal instantiation

D = set of animals   E(x) = "x eats food"

All animals eat food.
Jim is an animal
∴ Jim eats food

$\forall x \in D, E(x)$
Jim $\in D$
$\therefore E(\text{Jim})$

D = set of students   CS(x) = "x is in CS"
LC(x) = "x likes computers"

All CS students like computers. ✓ $\forall x \in D, \underbrace{CS(x) \to LC(x)}$,
Mika is a CS student   CS(Mika) ✓✓
→ Mika is a student ✓
∴ Mika likes computers

universal modus ponens = univ. inst. + modus ponens

$\therefore CS(\text{Mika}) \to LC(\text{Mika})$ ✓✓   univ inst
$LC(\text{Mika})$ modus ponens

CS(x) = "x is in CS"
LP(x) = "x likes pizza"

Nyah is a student   Nyah $\in D$
Nyah is a CS student   CS(Nyah)
Nyah likes pizza   LP(Nyah)
Nyah is a CS student who likes pizza $\boxed{CS(\text{Nyah}) \wedge LP(\text{Nyah})}$
∴ Some CS students like pizza.

$\exists x \in D$ s.t. $CS(x) \wedge LP(x)$

proof by example

P(x)
$x \in D$
$\exists x \in D$ s.t. P(x)

Some dogs play fetch $\exists x \in D, F(x)$
Find a dog x s.t. x plays fetch

(D: dogs; F(x): x plays fetch)

axiom of choice

Some CS instructors hate computers.

Jim is a CS inst.
Jim hates computers

Some perfect squares can be written as the sum of two non-zero perfect squares.

$25 = 9 + 16$

$5^2 = 3^2 + 4^2$

$\forall x \in \mathbb{R}, \; x > 3 \rightarrow x^2 > 9$

The square of any real number greater than 3 is greater than 9.

generalize from generic particular

Let $x \in \mathbb{R}$ [prove: $x > 3 \rightarrow x^2 > 9$]

$a, b, c, d > 0$

$a > b$

$c > d$

$ac > bd$

$x > 3$

$x > 3$

$x^2 > 9$

Suppose $x > 3$ [prove: $x^2 > 9$]

Then $x \cdot x > 3 \cdot 3 = 9$

$x^2 > 9$

$\rightarrow$ if $x > 3 \rightarrow x^2 > 9$

template for showing any real number $x > 3$ has $x^2 > 9$

$\forall x \in \mathbb{R}, \; x > 3 \rightarrow x^2 > 9$

$\mathbb{N}$ natural numbers $\{0, 1, 2, \ldots\}$

$\mathbb{Z}$ integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{R}$ real numbers

if $a, b \in \mathbb{Z}$, $a+b$ is an integer, $a \cdot b$ is an integer — integers closed under $+, \cdot$ (real numbers too)

For all a, b: $a+b=b+a$ and $a \cdot b = b \cdot a$ — commutative

axioms:

- a, b, c: $a+(b+c)=(a+b)+c$ and $a\cdot(b\cdot c)=(a\cdot b)\cdot c$ — associative
- $a\cdot(b+c)=ab+ac$ — distributive
- a: $a+0=0+a=a$ and $a\cdot 1 = 1\cdot a = a$ — identity
- there is an additive inverse $-a$ s.t. $a+(-a)=0$ — additive inverse
- if $a \neq 0$ then there is a multiplicative inverse $\frac{1}{a}$ s.t. $a\cdot\frac{1}{a}=\frac{1}{a}\cdot a = 1$ — mult inverse

$\forall a,b,c$ if $a+b=a+c$ then $b=c$ — cancellation

$\forall a,b\ \exists c$ s.t. $a+c=b$ $(c=b-a)$ — subtraction

$\forall a,b$ $a+(-b)=b-a$ — adding negative

$\forall a$ $-(-a)=a$, $a\cdot 0 = 0\cdot a = 0$ — double neg, mult by 0

$\forall a,b,c$ $a\cdot(b-c)=ab-ac$ — if $a\cdot b = a\cdot c$ and $a\neq 0$ then $b=c$ — cance

$\forall a,b$ if $a\cdot b=0$ then $a=0$ or $b=0$ — zero product

$a\cdot(-b)=(-a)\cdot b = -(ab)$ $(-a)(-b)=ab$ — mult of negatives

if $a,b>0$, then $a+b, a\cdot b > 0$ — closure of positives

$\forall a$ $a\neq 0 \to$ a is positive $\oplus$ $-a$ is positive

0 is not positive, 1 is positive

$a>b$ means there is a positive c s.t. $a=b+c$

$a<b$ means $b>a$

$\forall a,b$ exactly one of $a<b$, $b>a$, or $a=b$ is true — trichotomy

$\forall a,b,c$ $a<b$ and $b<c \to a<c$ — transitive

$a<b \to a+c<b+c$

$a<b$ and $c>0 \to ac<bc$ $a<b$ and $c<0 \to ac>bc$

$\forall a$, $a\neq 0 \to a\cdot a>0$

$\forall a,\ a \neq 0 \rightarrow a \cdot a > 0$

$\forall a,b \quad a \cdot b > 0 \rightarrow a,b > 0$ or $a,b < 0$ sign of product

$a > b \rightarrow -a < -b$ comparison of add. inv.

THM: For all real numbers a,b,c,d, if $a < b$ and $c < d$ then $a + c < b + d$

"n is even: $\exists k \in \mathbb{Z}$ s.t. $n = 2k$"

Def: For $n \in \mathbb{Z}$, n is even means there is an integer k s.t. $n = 2k$

n is odd means there is an integer k s.t. $n = 2k + 1$

n is odd: $\exists k \in \mathbb{Z}$ s.t. $n = 2k + 1$

THM: 0 is even

$\exists k \in \mathbb{Z}$ s.t. $0 = 2 \cdot k$

$0 = 2 \cdot 0$ (zero mult rule)

0 is an integer

there is an integer k (namely $k = 0$) s.t. $0 = 2k$

$\therefore$ 0 is even (def of even)

THM: For all integers a,b, if a and b are both even, then a+b is even.

$\forall a, b \in \mathbb{Z}$, a even $\wedge$ b even $\rightarrow$ a+b is even

Proof: Let $a, b \in \mathbb{Z}$ [want a even $\wedge$ b even $\rightarrow$ a+b even]

Suppose a is even and b is even [want: a+b even]

$\exists k \in \mathbb{Z}$ s.t. $a = 2k$ (def even)

let k be s.t. $k \in \mathbb{Z}$ and $a = 2k$ (choice)

$\exists \ell \in \mathbb{Z}$ s.t. $b = 2\ell$ (def even)

let ℓ be s.t. $\ell \in \mathbb{Z}$ and $b = 2\ell$

$a + b = 2k + 2\ell = 2(k + \ell)$ (sub, dist)

$k + \ell \in \mathbb{Z}$ (closure)

$\exists m \in \mathbb{Z}$ s.t. $a + b = 2m$ (ex' - $k+\ell$)

a+b is even (def even)

If a is even and b is even then a+b is even

$\forall a, b \in \mathbb{Z}$, if a,b even then a+b even (gen. from generic particular)

Divisibility

$b \nmid a$

b divides a
a is a multiple of b
b is a factor of a

$\exists k \in \mathbb{Z}$ s.t. $24 = 6 \cdot k$
$\exists k \in \mathbb{Z}$ s.t. $a = b \cdot k$

DEF: For integers a and b, $b \mid a$ means there is some integer k s.t. $a = b \cdot k$

$2 = 1 \cdot 2$ $\quad 24 = 6 \cdot 4$ $\quad 111 = 3 \cdot 37$

$1 \mid 2 \qquad 6 \mid 24 \qquad 37 \mid 111 \qquad 1 \mid 0$

$1 \mid 111$
$111 = 1 \cdot 111$

$0 = 1 \cdot 0$

$3 \mid 23$? NO $3 \nmid 23$

3716

$0 = 3716 \cdot k$

$\sim(\exists k \in \mathbb{Z}$ s.t. $23 = 3 \cdot k)$

$\forall k \in \mathbb{Z}, 23 \neq 3 \cdot k$

$a \mid 0 \quad \exists k \in \mathbb{Z}$ s.t. $0 = a \cdot k$

$\_ \mid 1496474$

$\forall a \in \mathbb{Z}, 1 \mid a$

THM: For every integer a, $1 \mid a$

Let $a \in \mathbb{Z}$. [want: $1 \mid a$, iow $\exists k \in \mathbb{Z}$ s.t. $a = 1 \cdot k$]

Then $a = 1 \cdot a$ (identity)
So $\exists k \in \mathbb{Z}$ s.t. $a = 1 \cdot k$ (example $\therefore k = a$)
$1 \mid a$ (def $\mid$)

For all $a \in \mathbb{Z}$, $1 \mid a$ (gen. from generic particular)

$\forall a, b \in \mathbb{Z}\ (a > 0 \wedge b > 0) \to (a \mid b \to a \leq b)$
$\equiv \forall a, b \in \mathbb{Z}\ (a > 0 \wedge b > 0 \wedge a \mid b \to a \leq b)$

THM: For all integers a, b s.t. $a, b > 0$, if $a \mid b$ then $a \leq b$

Proof: Let $a, b \in \mathbb{Z}$
Suppose $a, b > 0$ and $a \mid b$
Then $\exists k \in \mathbb{Z}$ s.t. $b = k \cdot a$; find that k. (def $\mid$)

$k < 0 \vee k = 0 \vee k > 0$
$p \vee q \vee r$
$p \to c$
$\therefore \sim p$
$q \to c$
$\therefore \sim q$
$\therefore r$

Suppose $k < 0$.
Then $ka < 0$ (mult by negative)
and $b < 0 \Rightarrow\Leftarrow$ $b < 0 \wedge b > 0$ (substitution)
$\therefore k < 0 \to c$ (conclusion from supposition)
$\therefore k \geq 0$ (contradiction rule, negation of $<$)
Suppose $k = 0$
Then $ka = 0$ (mult by 0)
and $b = 0 \Rightarrow\Leftarrow$ $b = 0 \wedge b > 0$ (substitution)
$\therefore k = 0 \to c$ (concl from suppose)
$\therefore k \neq 0$ (contradiction rule)
$\therefore k > 0$ (elimination)
$\therefore k \geq 1$ (k is an integer)
$\therefore ka \geq a$ (mult by positive)
$\therefore b \geq a$ (substitution)
$\therefore a, b > 0$ and $a \mid b \to b \geq a$ (conclusion from supposition)
$\forall a, b \in \mathbb{Z}\ a, b > 0$ and $a \mid b \to b \geq a$ (generalization from generic particular)

$\forall a, b, c \in \mathbb{Z}, a \mid b \wedge b \mid c \to a \mid c$

THM: For all integers a, b, c, if $a \mid b$ and $b \mid c$ then $a \mid c$

Proof: Let $a, b, c \in \mathbb{Z}$ [want $a \mid b \wedge b \mid c \to a \mid c$]

Proof: Let $a, b, c \in \mathbb{Z}$ [want $a|b \wedge b|c \to a|c$]

Suppose $a|b$ and $b|c$

$\exists k \in \mathbb{Z}$ s.t. $b = a \cdot k$; find that k (def $|$; axiom choice)

$\exists l \in \mathbb{Z}$ s.t. $c = b \cdot l$; find that l (def $|$; axiom of choice)

So $c = a k \cdot l$ (substitution)

note $k \cdot l \in \mathbb{Z}$ (closure of $\mathbb{Z}$ under $\cdot$)

$\exists m \in \mathbb{Z}$ s.t. $c = a \cdot m$ (ex: $m = k \cdot l$)

$a|c$ (def $|$)

$a|b \wedge b|c \to a|c$ (conclusion of suppose)

$\forall a, b, c \in \mathbb{Z}\ a|b \wedge b|c \to a|c$ (generalization from generic particular)