CPSC 430 : Formal Semantics
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Problem Set 4 Solution

Problem 3.1

1. $y = x^2$
2. $y = z \wedge x = w$
3. $(a \geq b \wedge y = x \times b) \vee (a < b \wedge y = x \times a)$
4. **true**
5. **false**

Problem 3.2

Consider the program:

```
newvar x:=a in
 newvar y:=b in
    (while y > 0 do
       newvar t:=x rem y in
          (x:=y; y:=t);
     c:=x)
```

The proof of total correctness is:

1. $(a \geq 0) \wedge (b \geq 0)$
   $\Rightarrow\ (a \geq 0) \wedge (b \geq 0)\ \wedge\ (gcd(a,b) = gcd(a,b))$
2. $[(a \geq 0) \wedge (b \geq 0)]$
   `x := a; y := b`
   $[(x \geq 0) \wedge (y \geq 0)\ \wedge\ (gcd(x,y) = gcd(a,b))]$ $(RAS_2, 1)$
3. $(x \geq 0) \wedge (y \geq 0)\ \wedge\ (gcd(x,y) = gcd(a,b)) \wedge (y > 0) \wedge (y = m)$
   $\Rightarrow\ (y \geq 0) \wedge (x\ \mathbf{rem}\ y \geq 0) \wedge (gcd(y, x\ \mathbf{rem}\ y) = gcd(a,b)) \wedge (x\ \mathbf{rem}\ y < m)$
4. $[(x \geq 0) \wedge (y \geq 0)\ \wedge\ (gcd(x,y) = gcd(a,b)) \wedge (y > 0) \wedge (y = m)]$
   `t := x rem y; x := y; y := t`
   $[(x \geq 0) \wedge (y \geq 0)\ \wedge\ (gcd(x,y) = gcd(a,b)) \wedge (y < m)]$ $(RAS_3, 3)$
5. $[(x \geq 0) \wedge (y \geq 0)\ \wedge\ (gcd(x,y) = gcd(a,b)) \wedge (y > 0) \wedge (y = m)]$
   `newvar t:=x rem y in (x:=y; y:=t)`
   $[(x \geq 0) \wedge (y \geq 0)\ \wedge\ (gcd(x,y) = gcd(a,b)) \wedge (y < m)]$ $(DC, 4)$
6. $(x \geq 0) \wedge (y \geq 0) \wedge (gcd(x,y) = gcd(a,b)) \wedge (y > 0)$
   $\Rightarrow\ (y \geq 0)$
7. $[(x \geq 0) \wedge (y \geq 0) \wedge (gcd(x,y) = gcd(a,b))]$
   `while y>0 do newvar t:=x rem y in (x:=y; y:=t)`
   $[(x \geq 0) \wedge (y \geq 0) \wedge (gcd(x,y) = gcd(a,b)) \wedge \neg(y > 0)]$ $(WHT, 5, 6)$
8. $(x \geq 0) \wedge (y \geq 0) \wedge (gcd(x,y) = gcd(a,b)) \wedge \neg(y > 0)$
   $\Rightarrow (x = gcd(a,b))$
9. $[(x \geq 0) \wedge (y \geq 0) \wedge (gcd(x,y) = gcd(a,b))]$
   `while y > 0 do newvar t:=x rem y in (x:=y; y:=t)`
   $[x = gcd(a,b)]$ $(WC, 7, 8)$
10. $[x = gcd(a,b)]$ `c:=x` $[c = gcd(a,b)]$ (AS)
11. $[(a \geq 0) \wedge (b \geq 0)]$
    `x:=a; y:=b; while y > 0 do newvar t:=x rem y in (x:=y; y:=t); c:=x`
    $[c = gcd(a,b)]$ $(MSQ_3, 2, 9, 10)$
12. $[(a \geq 0) \wedge (b \geq 0)]$
    `x:=a; newvar y:=b; in (while y > 0 do newvar t:=x rem y in (x:=y; y:=t); c:=x)`
    $[c = gcd(a,b)]$ $(DC, 11)$
13. $[(a \geq 0) \wedge (b \geq 0)]$
    `newvar x:=a in newvar y:=b; in (while y > 0 do newvar t:=x rem y in (x:=y; y:=t); c:=x)`
    $[c = gcd(a,b)]$ $(DC, 12)$

Problem 3.4

1. $[x > 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge x = cnt]$
   (x := x - 1; y := y + x)
   $[x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge x + 1 = cnt]$ (RAS$_2$)

2. $x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge x \neq 0 \wedge x = cnt \quad \Rightarrow$
   $x > 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge x = cnt$

3. $x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge x + 1 = cnt \quad \Rightarrow$
   $x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge x < cnt]$

4. $[x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge x \neq 0 \wedge x = cnt]$
(x := x - 1; y := y + x)
$[x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge x < cnt]$ (WC,SP 1,2,3)

5. $x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge x \neq 0 \quad \Rightarrow$
$x \geq 0$

6. $[x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2]$
while x $\neq 0$ **do** (x := x - 1; y := y + x)
$[x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge \neg(x \neq 0)]$ (WHT 4,5)

7. $x \geq 0 \wedge x = x_0 \wedge y = y_0 \quad \Rightarrow$
$x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2$

8. $x \geq 0 \wedge y = y_0 + (x_0 - x) \times (x_0 + x - 1) \div 2 \wedge \neg(x \neq 0) \quad \Rightarrow$
$y = y_0 + x_0 \times (x_0 - 1) \div 2$

9. $[x \geq 0 \wedge x = x_0 \wedge y = y_0]$
while x $\neq 0$ **do** (x := x - 1; y := y + x)
$[y = y_0 + x_0 \times (x_0 - 1) \div 2]$ (SP, WC, 6,7,8)