Sets - Notation

$x \in S$	membership	$\{\}$	the empty set
$x \in! S$	$S = \{x\}$	$\mathbf{N}$	natural numbers
$S \subseteq T$	inclusion	$\mathbf{Z}$	integers
$S \subseteq^{\mathsf{fin}} T$	finite subset	$\mathbf{B}$	$= \{\mathbf{true}, \mathbf{false}\}$

$\{E \mid P\}$	set comprehension	
$S \cap T$	intersection	$= \{x \mid x \in S \text{ and } x \in T\}$ x is a bound variable
$S \cup T$	union	$= \{x \mid x \in S \text{ or } x \in T\}$
$S - T$	difference	$= \{x \mid x \in S \text{ and not } x \in T\}$
$\mathcal{P}\, S$	powerset	$= \{T \mid T \subseteq S\}$
m to n	integer range	$= \{x \mid m \leq x \text{ and } x \leq n\}$

Set Operations (Generalized)

$$\cup S \stackrel{\text{def}}{=} \{x \mid \exists T \in S.\, x \in T\}$$

$$\bigcup_{i \in I} S \stackrel{\text{def}}{=} \cup\{S \mid i \in I\}$$

$$\bigcup_{i=m}^{n} S \stackrel{\text{def}}{=} \bigcup_{i \in\, m \text{ to } n} S$$

$$\cup\{\} = \{\}$$

$$\cap S \stackrel{\text{def}}{=} \{x \mid \forall T \in S.\, x \in T\}$$

$$\bigcap_{i \in I} S \stackrel{\text{def}}{=} \cap\{S \mid i \in I\} \ldots$$

$$\bigcap_{i=m}^{n} S \stackrel{\text{def}}{=} \bigcap_{i \in\, m \text{ to } n} S$$

$\cap\{\}$ meaningless

Examples:

$$A \cup B = \cup\{A, B\}$$

$$\cup\left\{i \text{ to } (i+1) \mid i \in \{j^2 \mid j \in 1 \text{ to } 3\}\right\} = \{1, 2, 4, 5, 9, 10\}$$

{ {1,2}, {4,5}, {9,10} }

Relations

A relation ρ is a set of primitive pairs $[x, y]$. $\neq \{\{x\},\{x,y\}\}$

$$(\rho\ x\ y\ z\ \ldots)$$

$$\rho \text{ relates } x \text{ and } y \iff x\ \rho\ y \iff [x, y] \in \rho$$

$$\rho \text{ is an identity relation} \iff (\forall x, y.\ x\ \rho\ y \Rightarrow x = y)$$

$$\text{the identity on } S \quad I_S \stackrel{\text{def}}{=} \{[x, x] \mid x \in S\}$$

$$\text{the domain of } \rho \quad \text{dom}\ \rho \stackrel{\text{def}}{=} \{x \mid \exists y.\ x\ \rho\ y\}$$

$$\text{the range of } \rho \quad \text{ran}\ \rho \stackrel{\text{def}}{=} \{x \mid \exists y.\ y\ \rho\ x\}$$

$$\text{composition of } \rho \text{ with } \rho' \quad \rho' \cdot \rho \stackrel{\text{def}}{=} \{[x, z] \mid \exists y.\ x\ \rho\ y \text{ and } y\ \rho'\ z\}$$

$$\text{reflection of } \rho \quad \rho^\dagger \stackrel{\text{def}}{=} \{[y, x] \mid [x, y] \in \rho\}$$

Relations - Properties & Examples

$$(\rho_3 \cdot \rho_2) \cdot \rho_1 = \rho_3 \cdot (\rho_2 \cdot \rho_1)$$

$$\rho \cdot I_S \subseteq \rho \supseteq I_T \cdot \rho$$

$$\text{dom } I_S = S = \text{ran } I_S$$

$$I_T \cdot I_S = I_{T \cap S}$$

$$I_S{}^\dagger = I_S$$

$$(\rho^\dagger)^\dagger = \rho$$

$$(\rho_2 \cdot \rho_1)^\dagger = \rho_1{}^\dagger \cdot \rho_2{}^\dagger$$

$$\rho \cdot \{\} = \{\} = \{\} \cdot \rho$$

$$I_{\{\}} = \{\} = \{\}^\dagger$$

$$\text{dom } \rho = \{\} \Rightarrow \rho = \{\}$$

$$I_{\mathbf{N}} = \{[0,0],[1,1],[2,2],\ldots\}$$

$$< \; = \{[0,1],[0,2],[1,2],\ldots\}$$

$$\leq \; = \{[0,0],[0,1],[1,1],[0,2],\ldots\}$$

$$\geq \; = \{[0,0],[1,0],[1,1],[2,0],\ldots\}$$

$$< \; \subseteq \; \leq$$

$$< \cup I_{\mathbf{N}} = \; \leq$$

$$\leq \cap \geq \; = I_{\mathbf{N}}$$

$$< \cap \geq \; = \{\}$$

$$< \cdot \leq \; = \; <$$

$$\leq \cdot \leq \; = \; \leq$$

$$\geq \; = \; \leq^\dagger$$

Functions

A relation f is a function if

$$\forall x, x', x''. ([x, x'] \in f \text{ and } [x, x''] \in f) \Rightarrow x' = x''$$

If f is a function,

$$f\ x = y \iff f_x = y \iff f \text{ maps } x \text{ to } y \iff [x, y] \in f$$

I_S and $\{\}$ are functions.

If f and g are functions, then $g \cdot f$ is a function: $(g \cdot f)\ x = g(f\ x)$

$f^\dagger$ is not necessarily a function:

consider $f = \{[\mathbf{true}, \{\}], [\mathbf{false}, \{\}]\}$

f is an injection if both f and $f^\dagger$ are functions.

Functions - Notation

Typed abstraction: $\lambda x \in S.\, E \stackrel{\text{def}}{=} \{[x, E] \mid x \in S\}$

Defined only when E is defined for all $x \in S$

(consider $\lambda g \in \mathbf{N}.\, g\ 3$)

$I_S = \lambda x \in S.\, x$

$g \cdot f = \lambda x \in \text{dom}\ f.\, g(f\ x)$, if $\text{ran}\ f \subseteq \text{dom}\ g$.

Placeholder: E with a dash $(-)$ standing for the bound variable

$g\,(-)\,h \;=\; \lambda x \in S.\, (g\,(x))\,h \qquad - + 42 \;=\; \lambda x \in \mathbf{N}.\, x + 42$

Variation of a function f: $[f \mid x : y]\, z = \begin{cases} y, & \text{if } z = x \\ f\ z, & \text{otherwise} \end{cases}$

$$\text{dom}\,[f \mid x : y] \;=\; (\text{dom}\ f) \cup \{x\}$$

$$\text{ran}\,[f \mid x : y] \;=\; ((\text{ran}\ f) - \{z \mid [x, z] \in f\}) \cup \{y\}$$

Sequences

$$[f \mid x_1 : y_1 \mid \ldots \mid x_n : y_n] \stackrel{\text{def}}{=} [\ldots[f \mid x_1 : y_1] \ldots \mid x_n : y_n]$$

$$[x_1 : y_1 \mid \ldots \mid x_n : y_n] \stackrel{\text{def}}{=} [\{\} \mid x_1 : y_1 \mid \ldots \mid x_n : y_n]$$

$$\langle x_0, \ldots x_{n-1} \rangle \stackrel{\text{def}}{=} [0 : x_0 \mid \ldots n-1 : x_{n-1}]$$

$[\,] = \{\}$ — the empty function

$\langle\rangle = [\,] = \{\}$ — the empty sequence

$\langle x_0, \ldots x_{n-1} \rangle$ — an n-tuple

$\langle x, y \rangle$ — a (non-primitive) pair

$\text{dom}\ \langle x_0, \ldots x_{n-1} \rangle = 0 \text{ to } (n-1)$

$\langle x_0, \ldots x_{n-1} \rangle_i = x_i$ when $i \in 0 \text{ to } (n-1)$

(Sets of Functions:) Products

Let θ be an indexed family of sets (a function with sets in its range).

The Cartesian product of θ is

$$\prod \theta \stackrel{\text{def}}{=} \{f \mid \text{dom } f = \text{dom } \theta \text{ and } \forall i \in \text{dom } \theta.\, f\ i \in \theta\ i\}$$

$$\begin{aligned}
&\prod \langle \mathbf{B},\ \mathbf{B} \rangle \\
&\quad = \prod(\lambda x \in 0 \text{ to } 1.\, \mathbf{B}) \\
&\quad = \{[0 : \mathbf{true}, 1 : \mathbf{true}],\ [0 : \mathbf{true}, 1 : \mathbf{false}], \\
&\qquad [0 : \mathbf{false}, 1 : \mathbf{true}],\ [0 : \mathbf{false}, 1 : \mathbf{false}]\} \\
&\quad = \{\langle \mathbf{true}, \mathbf{true} \rangle,\ \langle \mathbf{true}, \mathbf{false} \rangle,\ \langle \mathbf{false}, \mathbf{true} \rangle,\ \langle \mathbf{false}, \mathbf{false} \rangle\}
\end{aligned}$$

More Products

$$\prod_{x \in T} S \stackrel{\text{def}}{=} \prod \lambda x \in T.\, S \qquad S_1 \times \ldots \times S_n \stackrel{\text{def}}{=} \prod_{i=1}^{n} \text{“}S_i\text{”}$$

$$\prod_{i=m}^{n} S \stackrel{\text{def}}{=} \prod_{i \in (m \textbf{ to } n)} S \qquad S^T \stackrel{\text{def}}{=} \prod_{x \in T} S$$

$$S^n \stackrel{\text{def}}{=} S^{0 \textbf{ to } (n-1)} = \underbrace{S \times \ldots \times S}_{n \text{ times}}$$

$$\prod \langle \mathbf{B}, \mathbf{B} \rangle = \mathbf{B} \times \mathbf{B} = \mathbf{B}^2$$

$$S^0 = S^{\{\}} = \{\langle\rangle\} = \{\{\}\}$$

Sums

The disjoint union (sum) of θ is

$$\sum \theta \stackrel{\text{def}}{=} \{\langle i, x\rangle \mid i \in \text{dom}\,\theta \text{ and } x \in \theta\, i\}$$

$$\sum_{x \in T} S \stackrel{\text{def}}{=} \sum \lambda x \in T.\, S \qquad S_1 + \ldots + S_n \stackrel{\text{def}}{=} \sum_{i=1}^{n} \text{“}S_i\text{”}$$

$$\sum_{i=m}^{n} S \stackrel{\text{def}}{=} \sum_{i \in (m \textbf{ to } n)} S \qquad T \times S = \sum_{x \in T} S$$

$$n \times S = (0 \textbf{ to } (n-1)) \times S = \underbrace{S + \ldots + S}_{n \text{ times}}$$

$$\begin{aligned} \mathbf{B} + \mathbf{B} &= \sum \langle \mathbf{B}, \mathbf{B} \rangle = \{\langle 0, \mathbf{true}\rangle, \langle 0, \mathbf{false}\rangle, \langle 1, \mathbf{true}\rangle, \langle 1, \mathbf{false}\rangle\} \\ &= 2 \times \mathbf{B} \end{aligned}$$

Relations Between Sets

ρ is a relation from S to T

$$\iff \rho \in S \underset{\text{REL}}{\longrightarrow} T$$

$$\iff \text{dom}\,\rho \subseteq S \text{ and } \text{ran}\,\rho \subseteq T.$$

Relation on S $\stackrel{\text{def}}{=}$ relation from S to S.

$$I_S \in S \underset{\text{REL}}{\longrightarrow} S$$

$$\rho \in S \underset{\text{REL}}{\longrightarrow} T \Rightarrow \rho^{\dagger} \in T \underset{\text{REL}}{\longrightarrow} S$$

For all S and T, $\{\} \in S \underset{\text{REL}}{\longrightarrow} T$

$$\{\} \in! \; S \underset{\text{REL}}{\longrightarrow} \{\}$$

$$\{\} \in! \; \{\} \underset{\text{REL}}{\longrightarrow} T$$

Total Relations

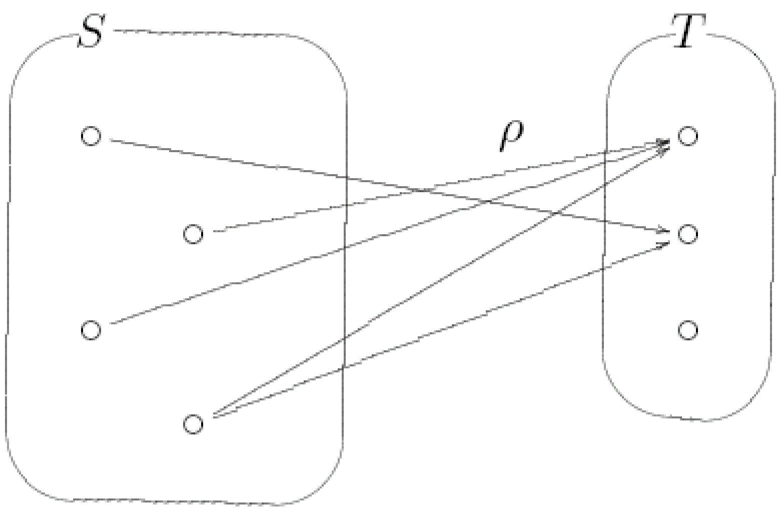
$\rho \in S \underset{\text{REL}}{\longrightarrow} T$ is a total relation from S to T

$$\iff \rho \in S \underset{\text{TREL}}{\longrightarrow} T$$

$$\iff \forall x \in S. \exists y \in T. x \; \rho \; y$$

$$\iff \text{dom } \rho = S$$

$$\iff I_S \subseteq \rho^\dagger \cdot \rho$$



$$\rho \in (\text{dom } \rho) \underset{\text{TREL}}{\longrightarrow} T \iff T \supseteq \text{ran } \rho$$

Functions Between Sets

f is a partial function from S to T

$$\iff f \in S \xrightarrow[\text{PFUN}]{} T$$

$$\iff f \in S \xrightarrow[\text{REL}]{} T \text{ and } f \text{ is a function.}$$

"Partial": $f \in S \xrightarrow[\text{REL}]{} T \;\Rightarrow\; \text{dom } f \subseteq S$

$f \in S \xrightarrow[\text{PFUN}]{} T$ is a (total) function from S to T

$$\iff f \in S \to T$$

$$\iff \text{dom } f = S.$$

- $S \to T = T^S = \prod_{x \in S} T$
- $S \to T \to U = S \to (T \to U)$

Surjections, Injections, Bijections

f is a surjection from S to $T \iff \text{ran } f = T$

f is a injection from S to $T \iff f^\dagger \in T \xrightarrow[\text{PFUN}]{} S$

f is a bijection from S to $T \iff f^\dagger \in T \rightarrow S$

$\iff f$ is an isomorphism from S to T