Population Protocols

James Aspnes
Yale University

January 29th, 2007
Acknowledgments

Joint work with:

- Dana Angluin (Yale)
- Melody Chan (Princeton)
- Zoë Diamadi (McKinsey & Company)
- David Eisenstat (Princeton)
- Michael J. Fischer (Yale)
- Hong Jiang (Yale)
- René Peralta (NIST)
- Eric Ruppert (York)
The past and future of computing

Economics of mass production push computer systems toward large numbers of very limited standardized components:

- Centralized systems
- Distributed systems
- Wireless distributed systems
- Sensor networks/RFID chips
- Smart molecules?

Our goal: take the limit of this process.

An interaction between two neighbors updates the state of both agents according to a joint transition function.

Interactions are asymmetric: one agent is the initiator and one the responder.

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Leader Election

- $\bullet$ = leader
- $\bullet$ = non-leader

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Interactions are asymmetric: one agent is the initiator and one the responder.
A **population protocol** (Angluin, Aspnes, Diamadi, Fischer, and Peralta, PODC 2004) consists of a collection of **finite-state agents** organized in an **interaction graph**.

An **interaction** between two neighbors updates the state of **both agents** according to a joint **transition function**.

Interactions are **asymmetric**: one agent is the **initiator** and one the **responder**.

**Leader Election**

- \[ \bullet, \bullet \rightarrow \bullet, \bullet \]
- \[ \bullet, \bullet \rightarrow \bullet, \bullet \]
- \[ \bullet, \bullet \rightarrow \bullet, \bullet \]
- \[ \bullet, \bullet \rightarrow \bullet, \bullet \]

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Leader Election

\[
\begin{align*}
\bullet, \bullet & \rightarrow \bullet, \bullet \\
\bullet, \bullet & \rightarrow \bullet, \bullet \\
\bullet, \bullet & \rightarrow \bullet, \bullet \\
\bullet, \bullet & \rightarrow \bullet, \bullet \\
\end{align*}
\]
Stable computations

- **Input map** converts inputs (at each agent) to initial states.
- **Output map** extracts outputs from states.
- A **stable computation** converges to the same output at all agents.
- **Fairness condition** enforces that any reachable state is eventually reached.

**Parity**

\[
\begin{align*}
\text{In:} & \\
0^*, 0^* & \rightarrow 0, 0^* \\
x & \rightarrow x^* \\
0^*, 1^* & \rightarrow 1, 1^* \\
1^*, 0^* & \rightarrow 1, 1^* \\
\text{Out:} & \\
x & \rightarrow x \\
1^*, 1^* & \rightarrow 0, 0^* \\
x^* & \rightarrow x \\
x^*, y & \rightarrow x, x^* \\
x, y^* & \rightarrow y^*, y
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**Out:**
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![Parity Diagram]
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1  1*
  /  \
1* 0*
```
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<td>$x^<em>, y \rightarrow x, x^</em>$</td>
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Jan 29, 2007

Population Protocols
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1

0*

0

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0

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```plaintext
Parity:

0
1

0*

0, 0* → 0, 0*
0*, 1* → 1, 1*
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![Parity Diagram](attachment:parity_diagram.png)
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![Parity State Transition Diagram](image)
Presburger predicates

- Trick: represent numbers by tokens scattered across the population.
- Population protocols on connected graphs can **stably compute** all of **first-order Presburger arithmetic** on counts of input tokens, including:
  - Addition.
  - Subtraction.
  - Multiplication by a constant $k$.
  - Remainder mod $k$.
  - $>$, $<$, and $=$.
  - $\land$, $\lor$, $\neg$, $\forall x$, and $\exists x$, applied to above.

- Example: “Are there at least twice as many cold sensors as hot sensors?”
Presburger predicates (continued)

- Computable for fixed inputs (Angluin et al., PODC 2004)
- Computable if inputs converge after some finite time (Angluin, Aspnes, Chan, Fischer, Jiang, and Peralta, DCOSS 2005).
- Computable with one-way communication (Angluin, Aspnes, Eisenstat, Ruppert, OPODIS 2005).
- Computable if a small number of agents fail (Delporte-Gallet, Fauconnier, Guerraoui, Ruppert, DCOSS 2006).
- Nothing else is computable on a complete interaction graph, i.e. if any agent can interact with any other (Angluin, Aspnes, Eisenstat, PODC 2006).
  - Example: can’t compute “Is the number of cold sensors the square of the number of hot sensors?”
What population protocols can’t do

- Complete interaction graph gives the *weakest* model.
- Conjectured in PODC 2004 paper that this model can only compute the Presburger predicates.
- Proved in PODC 2006 paper.
- We’ll describe a simplified version of this result now, then switch to what we *can* do in stronger models.
Presburger predicates in disguise

Other ways to define a Presburger predicate:

- Take a regular language $L$ and forget about the order of symbols in each word.
  - Resulting **Parikh map of a regular set** is Presburger-definable.
  - All Presburger-definable sets can be constructed this way.
  - Cute fact: going to context-free languages doesn’t change anything.

- Take a finite union of **linear sets** of the form

  $$\{ \vec{b} + k_1 \vec{x}_1 + k_2 \vec{x}_2 + \cdots + k_m \vec{x}_m \}.$$  

  - Resulting **semilinear set** is Presburger-definable.
  - All Presburger-definable sets can be constructed this way.
A semilinear set $S$, equal to the union of
- $\{(1,0) + k_1(1,0) + k_2(2,1)\}$ (dark circles), and
- $\{(0,2) + k_3(2,0)\}$ (shaded circles).
Plan

- Want to show that every predicate stably computed by a population protocol is a finite union of sets of the form
  \[ \{ \vec{b} + k_1\vec{x}_1 + k_2\vec{x}_2 + \cdots + k_m\vec{x}_m \} \].
- But the proof is too big to fit in the next eight slides.
- So instead we’ll prove a weaker **Pumping Lemma**: Every predicate stably computed by a population protocol is a finite union of **monoids**: sets of the form
  \[ \{ \vec{b} + k_1\vec{x}_1 + k_2\vec{x}_2 + \cdots \} \],
  where the number of terms may be infinite!
- This is the first step in the real proof, which then shows that finitely many terms are enough using large doses of algebra and geometry.
Higman’s Lemma

- **Higman’s Lemma:**
  - Any infinite sequence $a_1, a_2, \ldots$ in $\mathbb{N}^d$ has elements $a_i, a_j$ with $a_i \leq a_j$ and $i < j$.

- **Corollaries:**
  - Every subset of $\mathbb{N}^d$ has finitely many minimal elements. *(Dickson’s Lemma.)*
  - Every infinite subset of $\mathbb{N}^d$ contains an infinite ascending sequence $a_1 < a_2 < a_3 \ldots$. 
Recall a configuration is **output-stable** if you can’t generate a token with a different output.

If $x$ can generate a 1, so can any configuration $y \geq x$.

Non-stable configurations are closed upwards $\Rightarrow$ are union of cones over finitely many minimal points (by Dickson’s Lemma).

Non-stable configurations are semilinear $\Rightarrow$ so are stable configurations.
From preceding slide, \( x \) is output-stable iff \( x \not\geq b_i \) for some finite list of \( b_i \).

Let \( k > \max b_i(j) \) and define the **truncation**
\[
\tau_k(x(j)) = \min(k, x(j)).
\]

We can detect from \( \tau_k(x) \) if \( x \geq b_i \) or not.

\( \Rightarrow \) \( x \) is output-stable iff its truncation is.
Extensions

- Want to recognize when adding more tokens to an input doesn’t change its behavior.
- Define the set $X(c)$ of extensions of $c$ by

  $$X(c) = \{ x | \exists d : c + x \rightarrow d \text{ and } \tau_k(d) = \tau_k(c) \}.$$  

- Intuition is that $x$ is in $X(c)$ if $c$ can be “pumped” by $x$.
- Not hard to show that extensions are composable: if $x, y$ are in $X(c)$, then so is $x + y$. (This shows $\{c + X(c)\}$ is a monoid.)
Monoid covers

- Now we will hunt for a finite monoid cover of some stably computable $Y$ using extensions. The method is to build up a family of sets $x + X(c)$ where $x$ is an input and $c$ is an output-stable configuration reachable from that input.

- Order $Y$ so that $y_i \leq y_j$ implies $i < j$. Let $B_0 = \emptyset$. Compute $B_i$ as follows:
  - If $y_i \in x + X(c)$ for some $(x, c) \in B_{i-1}$, let $B_i = B_{i-1}$.
  - Otherwise, construct $B_i$ by adding to $B_{i-1}$ the pairs
    - $(y_i, s(y_i))$, and
    - $(y_i, s(c + y_i - x))$ for all $(x, c) \in B_{i-1}$ with $x \leq y_i$,
    where $s(z)$ is any stable configuration reachable from $z$.

- Finally, let $B = \bigcup B_i$. 

January 29th, 2007
B covers Y

Definition of $B_i$

- $B_i = B_{i-1}$ if $y_i \in x + X(c)$, $(x, c) \in B_{i-1}$; else
- $B_i = B_{i-1}$ plus
  - $(y_i, s(y_i))$, and
  - $(y_i, s(c + y_i - x))$ for all $(x, c) \in B_{i-1}$ with $x \leq y_i$.

1. $\{x + X(c)\}$ for $(x, c) \in B$ covers $Y$. (We add $y_i$ to $B_i$ if it doesn’t.)
2. $\{x + X(c)\}$ doesn’t contain anything outside $Y$. (Proof: $z \in x + X(c)$ implies $z \rightarrow z' \in c + X(c)$ and so $z$ converges to same output as $c$ by definition of $X(c)$.)
3. If we can show $B$ is finite, we are done.
Define $B_i$ as follows:

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1. Suppose $B$ is infinite.
2. Use Higman’s Lemma to get an increasing sequence $z_1 < z_2 < \ldots$ such that $(z_i, c_i) \in B$ for some $c_i$.
3. Use Higman’s Lemma again to get an infinite subsequence $(z_{i_j}, c_{i_j})$ where both $z$ and $c$ components are increasing.
4. Eventually, truncated $\tau_k(c_{i_{(j+1)}}) = \tau_k(c_{i_j})$.
5. But then $z_{i_{(j+1)}} - z_{i_j}$ is in $X(c_{i_j})$, so $z_{i_{(j+1)}}$ can’t be in $B$. 

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We just showed that any stably computable set has a finite cover by monoids:

\[ \{ \vec{b} + k_1 \vec{x}_1 + k_2 \vec{x}_2 + \ldots \}, \]

Corollary: Any infinite stably computable set \( S \) can be pumped: there is some \( \vec{b} \) and \( \vec{x} \) such that \( \vec{b} + k \vec{x} \) is in \( S \) for all \( k \in \mathbb{N} \).

Sadly, this is not enough to exclude some non-semilinear sets like \( \{(x, y) \mid x < y \sqrt{2} \} \).
Stably computable sets are semilinear, i.e. finite unions of sets of the form

\[ \{ \vec{b} + k_1 \vec{x}_1 + k_2 \vec{x}_2 + \cdots + k_m \vec{x}_m \}. \]

This excludes pretty much anything that requires multiplication, irrational constants, or nested loops to define.

Proof: see PODC 2006 paper.
Hooray! We’re done!

- Question: If we have an exact characterization of what population protocols can do, aren’t we done?
Hooray! We’re done!

- Question: If we have an exact characterization of what population protocols can do, aren’t we done?
- Answer: No.
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Answer: No.

- Bounded-degree interaction graph gives all of LINSPACE (Angluin et al., DCOSS 2005).
- Random scheduling in a complete graph gives all of LOGSPACE with exponential slowdown using simple techniques (Angluin et al., PODC 2004), or polylogarithmic slowdown using more sophisticated techniques (Angluin et al., DISC 2006).

Rest of talk: bounded-degree graphs, then random scheduling.
Leaders and followers

- Generate a single wandering leader token as in parity protocol.
- Leader deploys followers to mark out subgraphs.
- When two leaders collide, survivor cleans up extra followers.

Leader (●) obtains lower bound on degree by placing followers (○) on adjacent nodes.
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![Diagram showing leaders and followers in a graph](image)

**Computing degrees**

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![Diagram of graph with leaders and followers]

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Colors act as *local identifiers*, allowing a node to point to particular neighbors.
Distance-2 coloring

- Main problem is to detect duplicate colors among neighbors.
- Each node records how many times it has interacted with each neighbor (mod 2).
- On a mismatch, both nodes pick a new color nondeterministically.\(^a\)

\(^a\)Can be reduced to a deterministic protocol by exploiting nondeterministic scheduling—see OPODIS 2005 paper.

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Solution: build tree in parallel with coloring, reset tree builder whenever a node changes color.
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Distributed computation

- Unroll DFS traversal of spanning tree to get a linear-size Turing machine tape (Itkis and Levin, FOCS 1994).
- \(\Rightarrow\) bounded-degree graph can compute all of LINSPACE.
Is it practical?

Algorithms have poor performance even if we assume non-adversarial interaction pattern.

- Wandering leaders may require $\Theta(N^3)$ cover time to visit all nodes.
- Unique leader/colorizer/walker agents are bottlenecks.

More work is needed to get efficient algorithms and good programming tools.
Computation by epidemic

Last part:

- Back to complete interaction graph.
- But assume **random scheduling**.
- Goal: efficient computation in a test tube.
Randomized population protocols

- Assume next pair of agents to interact is chosen uniformly (i.e. with probability \( \frac{1}{N(N-1)} \)).
- This gives the randomized population protocol model from (Angluin et al., PODC 2004).
- It also is the uniform-rate case of the standard model for well-mixed chemical systems (e.g. (Gillespie 1977)).
- Expected **time** is obtained by dividing expected interactions by \( N \)—each agent interacts at a fixed rate regardless of size of the population.
A test-tube computer

- **Register values** (up to $O(N)$) are stored as tokens distributed across the population.
- A unique **leader agent** acts as the (finite-state) CPU.
- We want to support the usual operations of addition, subtraction, comparison, multiplication, division, etc.
- We want to do them all in polylogarithmic time ($O(N \log^{O(1)} N)$ interactions).
- We’ll accept a small ($O(N^{-\Theta(1)})$) probability of error.
Epidemics

- Key fact: An epidemic starting from one infected agent spreads to all agents in $\Theta(\log N)$ time with high probability.
- This gives us a broadcast primitive.
Instruction cycle

- Leader propagates a new opcode via epidemic.
- Followers carry out chosen operation:
  - $A \leftarrow 0$: Erase your $A$ token upon receipt of opcode.
  - $A \leftarrow A + B$: Make a new $A$ token for each $B$ token.
  - $A \not\equiv 0$: Start a counter-epidemic if you have an $A$.
  - $A > B$, $A \leftarrow A - B$, etc.: more complicated.
- Leader collects response (if any) from counter-epidemic, updates its state, and starts a new cycle.
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Problem: How does the leader know when to start the next instruction cycle?
Bounding the time for epidemics

- Average interactions to infect next victim is \( \frac{N(N-1)}{i(N-i)} \).
- For \( i > N/2 \), this is \( \Theta(N/i) \), the waiting time for coupon collector.
- \( \Rightarrow \) Known coupon collector concentration results (Kamath et al., 1995) bound \( i > N/2 \) case: \( \Theta(N \log N) \) w.h.p.
- Symmetry bounds \( i > N/2 \) case.
Phase clock

- Each agent is in a **phase** in the range 0 to $m - 1$.
- An initiator in a later phase mod $m$ recruits agents in earlier phases.
- The leader advances if it sees an initiator in its own phase.
- Result: Leader goes all the way around every $\Theta(\log N)$ time units.
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Phase clock: simulation results

Phase clock with $N = 1000$ and $m = 8$. 

January 29th, 2007
Phase clock: simulation results

Zoomed view of phase 0 and phase 4.
Why it works

- Phases $i$ and higher act as an epidemic wiping out phases $i - 1$ and lower.
- This epidemic finishes in $a \log N$ time (with high probability).
- When the leader advances, it takes at least $b \log N$ time (w.h.p.) to generate at least $N^\epsilon$ agents in the same phase $\Rightarrow$ leader advances before $b \log N$ time (a short phase) with probability $N^{O(\epsilon) - 1}$.
- For a sufficiently large number of phases $m$, the chance of too many short phases in a row is $O(N^{-c})$.
- Amazing fact: $m$ depends on $c$ but not $N$. 
Other operations

- Operations like assignment and addition that don’t require tokens to interact can be done in one instruction cycle ($O(\log N)$ time).
- Operations that do require interaction may take longer.
  - Naive $A > B$ algorithm: Have $A$ and $B$ tokens cancel until only one kind is left.
  - This takes $\Omega(N^2)$ interactions if there are few $A$’s and $B$’s.
- How can we do cancellation faster?
Cancellation by amplification

- Cancellation is fast if there are many tokens to cancel.
- Solution: Alternate between canceling and doubling.
- Invariant $A_k - B_k = 2^k(A_0 - B_0)$ after $k$ rounds.
- If no winner in $2\log N$ rounds, $A_0 = B_0$.
- This gives $A < B$ in $O(\log^2 N)$ time.
To compute $C \leftarrow A - B$, do binary search for $C$ such that $A = B + C$.

This takes $O(\log N)$ rounds of binary search at $O(\log^2 N)$ time each $\Rightarrow O(\log^3 N)$ time.

Similar approach for division gives $O(\log^4 N)$ time. (This is our most expensive operation.)
For a randomized population protocol with a unique initial leader, we have:

- **Register machine simulation:**
  - \( \Theta(\log N) \)-bit registers.
  - \( O(\log^4 N) \) expected time per operation.
  - \( O(N^{-c}) \) probability of failure.

- **Presburger predicate computation:**
  - \( O(\log^4 N) \) expected time. (Cf. \( O(N) \) for previous protocols.)
  - **Zero** probability of failure.
  - **Trick:** Combine fast fallible protocol with slow robust one.
What’s left?

- What happens if we don’t have a leader to start with?
  - Election by fratricide takes $\Theta(N^2)$ interactions.
  - Phase clock is irretrievably corrupted during election process.

- Can we elect a leader faster?
- Can we build a more robust phase clock?
- Can we cut down the polylog overhead?
Summary

What we have:

- Clean model of large-scale small-scale systems.
- Complete characterization of complete-graph case with adversarial scheduling.
- Powerful (but slow) Turing machine simulator for bounded-degree case with adversarial scheduling.
- Fast register-machine simulator for complete-graph case with random scheduling and an initial leader.
Summary

What we still want:

- Better algorithms and programming tools for bounded-degree case.
- Better understanding of intermediate large-degree cases.
- Better performance (and assumptions) for random-scheduling case.